MULTIPLE SOLUTIONS FOR PERTURBED INDEFINITE SEMILINEAR ELLIPTIC EQUATIONS

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Abstract
We are looking for infinitely many weak solutions for a semilinear elliptic equation with indefinite nonlinearity. The presence of an $L^2$ function perturbs the symmetry of the problem. The result is obtained using the approach introduced by Rabinowitz for positive nonlinearities.

1 Introduction
We are interested in the following problem

\[(P_f) \left\{ \begin{array}{l}
-\Delta u - \lambda u = W(x)p(u) + f(x) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega
\end{array} \right.\]

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where $\Omega$ is a bounded open subset of $\mathbb{R}^N$ ($N \geq 3$) with a smooth boundary. The parameter $\lambda$ varies in the whole real line $\mathbb{R}$ and $f$ is an $L^2(\Omega)$ function. The function $W(x)$ is bounded in $\Omega$ and different from zero almost everywhere ("thin" zero set, see $(W)$), while $p(u)$ is a continuous, subcritical (w.r.t. the Sobolev embedding) and superlinear function (see (p1)-(p3)). Moreover we ask that $p$ is an odd function ((p4)).

Therefore, for $f \equiv 0$, problem $(P_f)$ becomes symmetric and thus problems like $(P_f)$ are called perturbed while the indefiniteness of the problem is due to the change of sign of $W(x)$.

In recent years much work has been devoted to such indefinite problems (not only with symmetry): see e.g. [2], [3], [1], [9], [10], [17], [22], [20] for an idea of the results on this subject.

The main result of the paper, that is Theorem 3.2, assures that, under suitable assumptions, the energy functional $I_f$ associated with problem $(P_f)$ has an unbounded sequence of critical values provided

$$\theta = \frac{(N + 2) - (N - 2)s}{N(s - 1)} > \frac{s + 1}{s}$$

(1.1)

where $s$ is the growth exponent of the function $p$ appearing in $(p2)$.

Let us remind that, if $W(x)$ is a positive function, problem $(P_f)$ has been studied by Struwe [21], Bahri-Berestycki [7] and Rabinowitz [19]. The results contained in these three papers are slightly different but the interesting aspect is due to the utilization of three different approaches and methods of proof. In particular, Rabinowitz, in [19], proves that, if $\lambda = 0$ and $W(x) > 0$, $I_f$ has an unbounded sequence of critical values provided $s$ satisfies (1.1).

The indefinite perturbed problem $(P_f)$ has been studied in 1996 by Tehrani [23] and by Badiale [5] in case that $W(x)$ is a continuous changing sign function and has a "thick" zero set. More precisely they study problems of the following type

$$\begin{cases} -\Delta u - \lambda u = W(x)p(u) + f(x)h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

(1.2)

Using Struwe's approach, Tehrani proves that, if $W \equiv f$ and under suitable assumptions on $W, p, h$ if $\lambda$ does not belong to the spectrum of the Laplacian, problem (1.2) has infinitely many solutions provided $s$ satisfies (1.1). On the other hand, using the method of Bahri-Berestycky, Badiale achieves the same result for any $f \in L^2(\Omega)$ but only for $\lambda$ less than the first eigenvalue of the Laplacian and for a smaller range of $s$.

We observe that the class of functions $W$ we deal with is complementary to their (w.r.t. zero sets).
Let us explain now how we prove Theorem 3.2 following the ideas of Rabinowitz [10]. First of all, let us consider the symmetric problem

\[
\begin{align*}
-\Delta u - \lambda u &= W(x)p(u) \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega
\end{align*}
\]

where \( p \) and \( W \) are the same as in \((P_f)\). Multiplicity results for this problem were obtained, for example, in [2], [23], if \( \lambda \) does not belong to the spectrum of the Laplacian, under different assumptions w.r.t. ours, in particular the function \( W(x) \) is a continuous changing sign function and has a "thick" zero set.

Using a \( \mathbb{Z}_2 \) version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [4], we prove that for any real parameter \( \lambda \) problem \((P)\) possesses an unbounded sequence of weak solutions. We underline that we prove that the energy functional \( I \) associated with \((P)\) has an unbounded sequence \( \beta_k = I(u_k) \) of critical values and also that the sequence \( \{u_k\} \) of weak solutions is unbounded.

Therefore, in order to study problem \((P_f)\), we introduce an auxiliar \( C^1 \) functional \( J_f \). We show that, for large critical levels, the critical points of \( J_f \) are the same of \( I_f \) and \( J_f \) satisfies (PS) condition (3° and 4° of Proposition 3.5). So, to obtain our thesis, it is sufficient to show that \( J_f \) possesses an unbounded sequence of critical values.

The main difficulty to overcome was that of proving the properties of the functional \( J_f \) in the presence of a changing sign term. For this purpose we imposed an additional integral condition involving \( W \) and \( P \) (see \((pW)\)).

It is an interesting open question as to whether Theorem 3.2 holds weakening \((1.1)\). In the definite case Bahri Lions [8] and Bolle Ghoussoub Tehrani [11] were able to value of \( s \) to \( s < n(n-2)^{-1} \).

2 The symmetric case

Let us consider the following problem:

\[
\begin{align*}
-\Delta u - \lambda u &= W(x)p(u) \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega
\end{align*}
\]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^N \), \( N \geq 3 \), with smooth boundary \( \partial \Omega \), \( \lambda \in \mathbb{R} \). Let \( p \) be a nonlinear function satisfying the following assumptions:

\[ (p1) \quad p \text{ is a continuous function} \]

\[ (p2) \quad \exists a_1, a_2 > 0 : |p(t)| \leq a_1 + a_2 |t|^s, \quad s \in \left(1, \frac{N+2}{N-2}\right), \forall t \in \mathbb{R} \]

\[ (p3) \quad \exists r > 0 : 0 < (s+1)P(t) \leq tp(t) \quad \forall |t| \geq r, \text{ where } P(t) = \int_0^t p(\tau) \, d\tau. \]

Let us observe that hypothesis \((p3)\) implies that there exist two constants...
\[ a_4, a_5 > 0 \text{ such that} \]
\[ P(t) \geq a_5|t|^{s+1} - a_4 \quad \forall t \in \mathbb{R}. \]  
\tag{2.1}

Therefore there is a constant \( a_3 > 0 \) such that

\[ \frac{1}{s+1}(tp(t) + a_3) \geq P(t) + a_4 \geq a_5|t|^{s+1} \quad \forall t \in \mathbb{R}. \]  
\tag{2.2}

Furthermore let us assume that

\((p4)\quad p\) is an odd function.

Let \( W \in L^\infty(\Omega) \) be a function that assumes both positive and negative values in \( \Omega \), and let us define \( W^+(x) = \sup\{W(x), 0\} \) and \( W^-(x) = \inf\{W(x), 0\} \) for any \( x \in \Omega \) and \( W^\pm = \sup\{W^\pm(x) : x \in \Omega\} \). Let \( 0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots \) be the sequence of eigenvalues of the operator \(-\Delta\) with respect to the zero boundary conditions on \( \Omega \). Each eigenvalue \( \lambda_k \) is repeated according to its (finite) multiplicity. Let us require:

\[(pW1)\quad \exists R > 0: W^-(pt - (s + 1)P(t)) \leq \gamma |t|^2 \quad \forall t \geq R,\]

for some \( \gamma \in (0, \left(\frac{s+1}{2}-1\right)(\overline{X} - \lambda)) \),

with \( \overline{X} = \lambda_{k+1} \) if \( \lambda \in [\lambda_k, \lambda_{k+1}) \) \( k \in \mathbb{N} \), and \( \overline{X} = \lambda_2 \) if \( \lambda < \lambda_1 \)

\[(W)\quad \text{meas}\{x \in \Omega: W(x) = 0\} = 0, \; W^+(x) \neq 0.\]

The main result of this section is the following:

**Theorem 2.1** Suppose that the functions \( p, W \) satisfy \((p1) - (p4), (pW1), (W)\). Then problem \((P)\) possesses an unbounded sequence of weak solutions.

**Proof**

The proof of this theorem is an application of the following \(\mathbb{Z}_2\) version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz (H):

\[ I \in C^1(E, \mathbb{R}) \text{ be even, satisfy Palais–Smale condition, and } I(0) = 0. \]

\[ \text{Let } E^+, E^- \subset E \text{ be closed subspaces of } E \text{ with } \dim E^- - \text{codim } E^+ = 1 \text{ and suppose there holds:} \]

\[(I_1)\quad \exists R > 0: |I|_{|B_R \cap E^+} \geq \alpha,\]

\[(I_2)\quad \exists R > 0: I \leq 0 \text{ on } E^- \setminus B_R.\]

Consider the following set:
Γ = \{ h \in C^0(E, E) : h \text{ is odd, } h(u) = u \text{ if } u \in E^-, ||u|| \geq R \}

Then

(a) \forall \delta > 0, h \in \Gamma, \partial B_\delta \cap E^+ \cap h(E^-) \neq \emptyset

(b) The number \( \beta := \inf_{h \in \Gamma} \sup_{u \in E^-} I(h(u)) \geq \alpha \)

is a critical value for \( I \).

Let \( E = H^1_0(\Omega) \), equipped with the norm

\[ ||u|| = \left( \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}, \]

and

\[ I(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} u^2 - W(x) P(u). \tag{2.3} \]

Obviously \( I \in C^1(E, \mathbb{R}) \), \( I(0) = 0 \) and is even. Moreover, (PS) condition holds for any \( \lambda \in \mathbb{R} \) by \((pW1)\) and \((W)\), as it was proved in [2, 15], if \( \lambda < \lambda_1 \), and in [14] if \( \lambda \geq \lambda_1 \).

To verify \((I_1)\) let us suppose \( \lambda \geq 0 \), as the case \( \lambda < 0 \) can be proved in a simpler way. For any \( \lambda \in \mathbb{R}^+ \), \( \lambda \in [\lambda_{k_0-1}, \lambda_{k_0}] \neq \emptyset \), where we assume \( \lambda_0 = 0 \). Let us choose \( V_{k_0-1} = \langle v_1, \ldots, v_{k_0-1} \rangle \) where the function \( v_j \) is an eigenfunction associated to \( \lambda_j \) and \( E^+ = E_0^+ = V_{k_0-1}^\perp \), hence \( \text{codim} E_0^+ = k_0 - 1 \). By \((p2)\), for any \( u \in E \),

\[ I(u) \geq \frac{1}{2} \left( \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \right) - CW^+ \int_\Omega |u|^{s+1} - C, \]

where \( C \) (here and in the following) denotes a generic positive constant.

By the Gagliardo–Nirenberg inequality (see e.g. [16], [13])

\[ ||u||_{L^{s+1}} \leq \alpha_s \left( \int_\Omega |\nabla u|^2 \right)^{\frac{s}{2}} \left( \int_\Omega |u|^2 \right)^{\frac{1-s}{2}}, \]

where \( a \in (0, 1) \) is defined by

\[ \frac{1}{s+1} = a \left( \frac{1}{2} - \frac{1}{N} \right) + (1 - a) \frac{1}{2}, \]

and, by the variational characterization of the eigenvalues,

\[ \left( \int_\Omega |u|^2 \right)^{\frac{1}{2}} \leq \lambda_{k_0}^{-\frac{1}{2}} \left( \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}, \forall u \in E_0^+, \]
one gets, for any \( u \in E_0^+ \cap \partial B_\rho \),
\[
I(u) \geq \rho^2 - CW^+ \left( \alpha^{s+1} \|u\|^{s+1} \lambda_k \right) - \frac{\lambda}{2} \lambda_k^{-1} \|u\|^2 - C
\]
(2.4)
\[
\geq \rho^2 \left( \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) - C \lambda_k \left( \frac{1}{2} \right)^{s-1} \rho^{s-1} \right) - C
\]
which easily implies (I.).

Let us prove (I_2). We choose \( k_0 \) mutually disjoint balls \( B_1, \ldots, B_{k_0} \), where \( B_i \subset \Omega^+ \) and \( \Omega^+ := \{ x \in \Omega : W(x) > 0 \} \), in such a way that \( \Omega^+ \cup \bigcup_{i=1}^{k_0} B_i \neq \emptyset \), and \( k_0 \) nonnegative functions \( \varphi_1, \ldots, \varphi_{k_0}, \varphi \in C_0^\infty (B_i) \) for \( i = 1, \ldots, k_0 \). Then set
\[
E^- = E_0^- := \text{span}\{ \varphi_1, \ldots, \varphi_{k_0} \}.
\]
Therefore \( \dim E_0^- - \text{codim} E_0^+ = 1 \). Let \( u \in E_0^- \setminus B_{R_0} \) for \( R_0 \) sufficiently large, hence \( u = \sum_{i=1}^{k_0} t_i \varphi_i, \ t_i \in \mathbb{R} \). For any \( \lambda \in \mathbb{R}^+ \), by (2.1),
\[
I(u) = \frac{1}{2} \left( \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \right) - \int_{\Omega^+} W^+(x)P(u)
\]
\[
\leq C \frac{1}{2} \sum_{i=1}^{k_0} |t_i|^2 \|\varphi_i\|^2 - C \sum_{i=1}^{k_0} |t_i|^{s+1} \int_{\Omega^+} W^+(x)|\varphi_i|^{s+1} + C
\]
(2.5)
\[
\leq C \frac{1}{2} \sum_{i=1}^{k_0} |t_i|^2 - C \left( \sum_{i=1}^{k_0} |t_i|^2 \right)^{s+1} + C.
\]
Therefore \( I(u) < 0 \) as \( s + 1 > 2 \) and \( R_0 = C \sum_{i=1}^{n} |t_i|^2 \rightarrow +\infty \).

Now Theorem 2.2 applies and we get a critical value
\[
\beta_0 := \inf_{h \in \Gamma_0} \sup_{u \in E_0^-} I(h(u))
\]
where
\[
\Gamma_0 = \{ h \in C^0 (E, E) : h \text{ is odd, } h(u) = u \text{ if } u \in E_0^-, \|u\| \geq R_0 \}.
\]

Next we repeat the same procedure now working with
\[
E_1^+ = V_{k_0}^+.
\]
and

\[ E_1^- = \text{span}\{\varphi_1, \ldots, \varphi_{k_0+1}\}, \]

with \( \varphi_{k_0+1} \in C_0^\infty(B_{k_0+1}) \), the ball \( B_{k_0+1} \subset \Omega^+ \) being disjoint from \( B_i \) for \( i = 1, \ldots, k_0 \) and \( \Omega^+ \setminus \bigcup_{i=1}^{k_0+1} \neq \emptyset \). We find a second critical value

\[ \overline{\beta}_1 := \inf_{h \in \Gamma_1} \sup_{u \in E_1^-} I(h(u)) \]

where \( \Gamma_1 \) is defined as \( \Gamma_0 \), with \( R_0 \) replaced by \( R_1 > R_0 \). Continuing in this fashion, we find infinitely many critical values \( \overline{\beta}_k, k \geq 0 \).

Now we want to obtain an estimate from below for \( \overline{\beta}_k \). By (a) there exists \( w \in \partial B_\rho \cap E_1^+ \cap h(E_1^-) \), therefore

\[ \sup_{u \in E_1^-} I(h(u)) \geq I(w) \geq \inf_{u \in \partial B_\rho \cap E_1^+} I(u). \]

To obtain a lower bound for the right hand side of (2.6), we observe that, by (2.4) since \( \left( 1 - \frac{\lambda}{\lambda_{k_0}} \right) > 0 \) we can choose \( \rho = \rho(k_0) \) such that the coefficient of \( \rho^2 \) is 1/4 that is

\[ \left( \frac{1}{2} - \frac{\lambda}{\lambda_{k_0}} \right) - C\lambda_{k_0}^{\frac{(s+1)(s-1)}{2}} \rho^{s-1} = \frac{1}{4}. \]

First of all this implies

\[ \rho(k_0)^{s-1} = C\lambda_{k_0}^{\frac{(1-a)(s+1)}{2}} \]

hence for any \( k \geq 0 \)

\[ \rho(k_0 + k)^{s-1} = C\lambda_{k_0+k}^{\frac{(1-a)(s+1)}{2}}. \]

Moreover, by (2.7) and (2.4), one finds that

\[ I(u) \geq \frac{1}{4} \rho(k_0 + k)^2 - C \]

for any \( u \in \partial B_\rho \cap E_1^+ \). Since \( \lambda_j \geq Cj^{\frac{2}{s}} \) for large \( j \) (see [12]), we obtain

\[ \overline{\beta}_k \geq C\lambda_{k_0+k}^{\frac{(1-a)(s+1)}{2}} \geq Ck^{\frac{(1-a)(s+1)}{2(s-1)}} \]

for \( k \) large.

We have proved that \( I \) possesses an unbounded sequence of critical values \( \overline{\beta}_k = I(u_k) \) where \( u_k \) is a weak solution of (P).

Now we are going to prove that \( \{u_k\} \) is unbounded in \( E \). Since \( I'(u_k)u_k = 0 \), for any \( \lambda \in \mathbb{R} \) we have

\[ \int_{\Omega} |\nabla u_k|^2 = \int_{\Omega} (\lambda u_k^2 + W(x)p(u_k)u_k) \geq 0 \]

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hence
\[ \int_\Omega W^+(x)p(u_k)u_k + \int_\Omega \lambda u_k^2 \geq \int_\Omega W^-(x)p(u_k)u_k. \] (2.8)

Let us observe that, by (p2) and (2.1)
\[ \int_\Omega W^+(x)p(u_k)u_k \leq a_1 \int_\Omega W^+(x)|u_k| + a_2 \int_\Omega W^+(x)|u_k|^{s+1} \]
\[ \leq a_1 W^+ ||u_k||_{L^2}^2 + \frac{a_2}{a_5} \int_\Omega W^+(x)P(u_k) + C. \] (2.9)

Therefore, by (2.8), (2.9), (p3)
\[ (\lambda + a_1 W^+)||u_k||_{L^2}^2 + \frac{a_2}{a_5} \int_\Omega W^+(x)P(u_k) \]
\[ \geq (s + 1) \int_\Omega W^-(x)P(u_k) - C \] (2.10)

that is
\[ C_1 ||u_k||_{L^2}^2 + \int_\Omega W(x)P(u_k) \geq -C \] (2.11)

where \( C_1 = (\lambda + a_1 W^+)\left(\max \frac{a_2}{a_5}, s + 1\right)^{-1} \). So (2.11) and the fact that \( \overline{J}_k = \frac{1}{2} \int_\Omega (W(x)p(u_k)u_k - 2W(x)P(u_k)) \to +\infty \)

yield
\[ \max (1, 2C_1 - \lambda) ||u_k||^2 \geq \int_\Omega |\nabla u_k|^2 + (2C_1 - \lambda) \int_\Omega u_k^2 \]
\[ = \int_\Omega W(x)p(u_k)u_k + \int_\Omega \lambda u_k^2 + (2C_1 - \lambda) \int_\Omega u_k^2 \]
\[ + \int_\Omega 2W(x)P(u_k) - \int_\Omega 2W(x)P(u_k) \]
\[ \geq -C + \int_\Omega (W(x)p(u_k)u_k - 2W(x)P(u_k)) \to +\infty. \]

Hence \( \{u_k\} \) must be unbounded in \( E \).
\[ \square \]
3 The nonsymmetric case

This section deals with \((P_f)\) where \(f \in L^2(\Omega), f \neq 0.\) The corresponding Euler functional is

\[ I_f(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} u^2 - W(x)P(u) - f(x)u \right) u \in H^1_0(\Omega) \quad (3.1) \]

Let us define the set

\[ I_\varepsilon = \{ u \in H^1_0(\Omega) : I'_f(u) = 0, \ I_f(u) \geq \varepsilon > 0 \} \]

where \(\varepsilon\) is a sufficiently small positive number. We assume the following hypothesis:

\[ (pW) \quad \exists \delta > 0 : 0 < \int_\Omega W^-(x)h(u_k) \int_\Omega W^+(x)h(u_k) \leq 1 - \delta, \quad \text{definitively} \quad \forall \ {u_k} \in I_\varepsilon \]

where \(h(u_k) = u_k p(u_k) - 2P(u_k).\)

**Lemma 3.1** If \((pW)\) holds, there exists a constant \(\tilde{C} > 0\) such that

\[ \tilde{C} \left| \int_\Omega W(x)h(u) \right| \geq \int_\Omega |W(x)|h(u) \quad \forall u \in I_\varepsilon \quad (3.2) \]

*Obviously the constant \(\tilde{C} \geq 1.\)*

**Proof**

By \((pW)\)

\[ 0 < \limsup_{\{k: u_k \in I_\varepsilon\}} \frac{\int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k)} = K < 1 \]

So

\[ \limsup_{\{k: u_k \in I_\varepsilon\}} \frac{\int_\Omega W^+(x)h(u_k) + \int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k) - \int_\Omega W^-(x)h(u_k)} \leq \frac{1 + \limsup_{\{k: u_k \in I_\varepsilon\}} \frac{\int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k)}}{1 + \liminf_{\{k: u_k \in I_\varepsilon\}} \left( -\frac{\int_\Omega W^- (x)h(u_k)}{\int_\Omega W^+(x)h(u_k)} \right)} = 1 + K \]

We are going to prove the following Theorem, which extends the result by Rabinowitz (see Theorem 10.4 of [13]) to the case of indefinite nonlinearities, following the same spirit:

**Theorem 3.2** If \(p\) and \(W\) satisfy \((p1) - (p4), (pW), (pW1), (W)\) and \(f \in L^2(\Omega), f \neq 0,\) then problem \((P_f)\) possesses infinitely many solutions, provided that \(s\) in \((p2)\) is further restricted by the condition \([1,7].\)
In order to prove Theorem 3.2 we will need some additional lemmas and propositions.

**Lemma 3.3** Under the hypotheses of Theorem 3.2 there exists a constant $A$ depending on $||f||_{L^2}$ such that, if $u$ belongs to $I_\varepsilon$, then

$$\int_\Omega |W(x)| (P(u) + a_4) \leq A \left( I_f(u)^2 + 1 \right)^{\frac{1}{2}} \quad (3.3)$$

**Remark 3.4** If $f \equiv 0$ and $P$ is a homogeneous function, one can show, without hypothesis $(pW)$, that

$$\int_\Omega W(x)|u|^{s+1} \leq A \left( I_f(u)^2 + 1 \right)^{\frac{1}{2}}. \quad (3.4)$$

This suggests that the estimate (3.3) maybe is too strong. However, the proofs carried out in the sequel will highlight that an estimate like (3.5) is not useful, because $W$ changes sign.

**Proof**

Suppose $u$ is a critical point of $I_f$, then by $(p3)$ and by Remark 3.1

$$|I_f(u)| = \left| I_f(u) - \frac{1}{2} I_f(u) u \right| = \left| \int_\Omega \frac{1}{2} W(x)p(u)u - W(x)P(u) - \frac{1}{2} fu \right|$$

$$\geq \frac{1}{2} \left| \int_\Omega W(x)(p(u)u - 2P(u)) \right| - \frac{1}{2} \left| \int_\Omega fu \right|$$

$$\geq \frac{1}{2C} \int_\Omega |W(x)|p(u)u - \frac{1}{C} \int_\Omega |W(x)|P(u) - \frac{1}{2} ||f||_{L^2} ||u||_{L^2} \quad (3.5)$$

$$\geq \frac{s-1}{2C} \int_\Omega |W(x)|P(u) - \frac{1}{2} ||f||_{L^2} ||u||_{L^2}$$

$$\geq \frac{s-1}{2C} \int_\Omega |W(x)|(P(u) + a_4) - C||u||_{L^2}^2 - C$$

Let us observe now that by the Hölder inequality,

$$||u||_{L^2}^2 = \int_{\text{supp } W} \frac{1}{|W(x)|^{\frac{s+1}{s}}} |W(x)|^{\frac{s+1}{s}} |u|^2 \leq C \left( \int_\Omega |W(x)||u|^{s+1} \right)^{\frac{s}{s+1}}. \quad (3.6)$$

By the Young inequality, since $s + 1 > 2$, for $\varepsilon > 0$,

$$C \left( \int_\Omega |W(x)||u|^{s+1} \right)^{\frac{s}{s+1}} \leq C(\varepsilon) + \varepsilon \int_\Omega |W(x)||u|^{s+1}$$
where $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Hence by (3.5) and using (2.2), one finds that

$$|I_f(u)| \geq \left( \frac{s-1}{2C} - \frac{\varepsilon}{a_5} \right) \int_\Omega |W(x)|(P(u) + a_4) - C(\varepsilon) - C \quad (3.7)$$

so, choosing $\varepsilon$ such that $\left( \frac{s-1}{2C} - \frac{\varepsilon}{a_5} \right) > 0$ the thesis follows. \hfill \square

Now let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\chi(\xi) \equiv 1$ for $\xi \leq 1$, $\chi(\xi) \equiv 0$ for $\xi \geq 2$, and $\chi'(\xi) \in (-2,0)$ for $\xi \in (1,2)$. Moreover let us define

$$Q(u) \equiv 2A \left( I_f(u)^2 + 1 \right)^{\frac{1}{2}}$$

and

$$\psi(u) \equiv \chi \left( Q(u)^{-1} \int_\Omega |W(x)|(P(u) + a_4) \right).$$

Observe that, by Lemma [3.3] if $u \in I_\varepsilon$, the argument of $\chi$ lies in $[0, \frac{1}{2}]$, therefore $\psi(u) \equiv 1$.

Finally set

$$J_f(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} u^2 - W(x)P(u) - \psi(u)f(x)u \right). \quad (3.8)$$

Then $J_f(u) = I_f(u)$ if $u$ is a critical point of $I_f$ with $I_f(u) \geq \varepsilon > 0$. In the next proposition we will show the main properties of $J_f$.

**Proposition 3.5** Under the hypotheses of Theorem 3.2 the following properties of $J_f$ hold:

1° $J_f \in C^1(E, \mathbb{R})$;

2° There exists a constant $D$ depending on $||f||_{L^2}$ such that

$$|J_f(u) - J_f(-u)| \leq D \left( |J_f(u)|^{\frac{1}{1+r}} + 1 \right) \quad \text{for all } u \in E;$$

3° There exists a constant $M_0 > 0$ such that if $J_f(u) \geq M_0$ and $J'_f(u) = 0$,

then $J_f(u) = I_f(u)$ and $I'_f(u) = 0$;

4° There exists a constant $M_1 \geq M_0$ such that, for any $c > M_1$,

$J_f$ satisfies $(PS)_{loc}$ at $c$. 

11
Proof
1° It is true by standard arguments
2° Observe that
\[
\left| \int_{\Omega} fu \right| \leq \alpha_1 \left( |I_f(u)|^{1/\tau} + 1 \right). \tag{3.9}
\]
Indeed, since \((W)\) holds, by (2.2), one gets
\[
\left| \int_{\Omega} fu \right| = \left| \int_{\text{supp } W} f \frac{W(x)^{-\frac{1}{\tau}}}{W(x)} W(x)^{\frac{1}{\tau}} u \right| \\
\leq \left| fW(x)^{-\frac{1}{\tau}} \right|_{L^2} \left| W(x)^{\frac{1}{\tau}} u \right|_{L^2} \leq C \left| W(x)^{\frac{1}{\tau}} u \right|_{L^{r+1}} \tag{3.10}
\]
Now, if \(u \in \text{supp } \psi\), one has
\[
\int_{\Omega} |W(x)|(P(u) + a_4) \leq 2Q(u) = 4A \left( |I_f(u)|^2 + 1 \right)^{\frac{1}{2}} \leq C \left( |I_f(u)| + 1 \right),
\]
so the last inequality and (3.10) yield (3.9).
Now let us estimate the “lack” of symmetry of \(J_f\).
\[
|J_f(u) - J_f(-u)| = \left| \int_{\Omega} (-\psi(u)fu + \psi(-u)f(-u)) \right| \\
\leq \left( \psi(-u) + \psi(u) \right) \left| \int_{\Omega} fu \right|. \tag{3.11}
\]
Observe that, by (3.9), one has
\[
\psi(u) \left| \int_{\Omega} fu \right| \leq \psi(u)\alpha_1 \left( |I_f(u)|^{1/\tau} + 1 \right), \tag{3.12}
\]
so, by the definitions of \(I_f\), \(J_f\), and by (3.12), we deduce
\[
|I_f(u)| = \left| \int_{\Omega} \frac{1}{2} |\nabla u|^2 - \frac{\lambda u^2}{2} - W(x)P(u) - fu - \psi fu + \psi fu \right| \\
\leq \left| J_f(u) \right| + |\psi - 1| \left| \int_{\Omega} fu \right| \leq \left| J_f(u) \right| + \left| \int_{\Omega} fu \right|. \tag{3.13}
\]
So, by (3.12), (3.13) and using the Young inequality,
\[
\psi(u) \left| f_\Omega f u \right| \leq C \psi(u) \left( |J_f(u)|^{\frac{1}{s+1}} + \left| \int_\Omega f u \right|^{\frac{1}{s+1}} + 1 \right)
\]
(3.14)

Hence, using (3.14) and a similar estimate for \( \psi(-u) \), together with (3.11), yields 2°.

To prove 3° it suffices to prove that for \( M_0 \) large, if \( u \) is critical for \( J_f \) with \( J_f(u) \geq M_0 \), then

\[
Q(u)^{-1} \int_{\Omega} |W(x)| (P(u) + a_4) < 1.
\]
(3.15)

The definition of \( \psi \) then implies \( \psi(v) \equiv 1 \) for \( v \) near \( u \). Hence \( \psi'(u) = 0 \), so \( J_f(u) = I_f(u) \), \( J'_f(u) = I'_f(u) \) and 3° will follow.

Actually by the definition of \( J_f \) one gets

\[
J'_f(u)u = \int_{\Omega} \left[ |\nabla u|^2 - \lambda u^2 - W(x)p(u)u - (\psi(u) + \psi'(u)u)f u \right].
\]
(3.16)

Setting \( \theta(u) = Q(u)^{-1} \int_{\Omega} |W(x)| (P(u) + a_4), \) one has

\[
\psi'(u)u = \chi'(\theta(u))Q(u)^{-2} \left[ Q(u) \int_{\Omega} |W(x)p(u)u - uQ'(u) \int_{\Omega} |W(x)| (P(u) + a_4) \right]
\]

\[
= \chi'(\theta(u))Q(u)^{-2} \left[ Q(u) \int_{\Omega} |W(x)p(u)u - (2A)^2 I_f(u)I'_f(u)u \theta(u) \right]
\]

Set now

\[
T_1(u) = \chi'(\theta(u))(2A)^2 Q(u)^{-2} I_f(u)\theta(u) \int_{\Omega} f u
\]

\[
T_2(u) = \chi'(\theta(u))Q(u)^{-1} \int_{\Omega} f u.
\]

One gets

\[
J'_f(u)u = (1 + T_1(u)) \int_{\Omega} |\nabla u|^2 - \lambda(1 + T_1(u)) \int_{\Omega} u^2
\]

\[
-(1 + T_1(u)) \int_{\Omega} W(x)p(u)u - T_2 \int_{\Omega} |W(x)|p(u)u
\]

\[
-(\psi(u) + T_1(u)) \int_{\Omega} f u.
\]
(3.17)
Consider now

\[ J_f(u) - \frac{1}{2(1 + T_1(u))} J'_f(u)u \]  

(3.18)

If \( \psi(u) = 1 \) and \( T_1(u) = T_2(u) = 0 \) then (3.18) reduces to the left-hand side of (3.5), so (3.15) is a consequence of (3.3). Since \( 0 \leq \psi(u) \leq 1 \), if \( T_1(u) \) and \( T_2(u) \) are small enough, the calculation made in (3.5) can be used in (3.18) to prove a similar estimate, with \( A \) replaced by a larger constant, still smaller than \( 2A \), so again (3.15) holds. Therefore it suffices to show that \( T_1(u), T_2(u) \rightarrow 0 \) as \( M_0 \rightarrow \infty \). If \( u \notin \text{supp} \psi \) then \( T_1(u) = T_2(u) = 0 \), so we assume \( u \in \text{supp} \psi \). By (3.9) and the definition of \( T_1, \chi \),

\[
|T_1(u)| \leq 2(2A)^2 \frac{|I_f(u)|}{(2A)^2|I_f(u)|^2 + 1} \int_\Omega f u \\
\leq 4\alpha_1 |I_f(u)|^{-1} \left( |I_f(u)|^{\frac{1}{\nu}} + 1 \right)
\]

(3.19)

Now, if \( u \in \text{supp} \psi \), by the definition of \( J_f \) and \( I_f \) one gets

\[ I_f(u) \geq J_f(u) - \left| \int_\Omega f u \right|. \]

Thus, by (3.9)

\[ I_f(u) + \alpha_1 |I_f(u)|^{\frac{1}{\nu}} \geq J_f(u) - \alpha_1 \geq \frac{M_0}{2} \]

(3.20)

for \( M_0 \) to be chosen large enough. Therefore, if \( I_f(u) \leq 0 \), (3.20) and the Young inequality with \( (s + 1)' = \nu \) imply

\[
\frac{\alpha_1^{s'}}{\nu} + \frac{|I_f(u)|}{s + 1} \geq \frac{M_0}{2} + |I_f(u)|
\]

which is impossible if we choose \( M_0 \geq \frac{2\alpha_1^{s'}}{\nu} \). Hence \( I_f(u) \geq 0 \). Therefore (3.20) yields \( I_f(u) \geq \frac{M_0}{4} \) or \( I_f(u) \geq \left( \frac{M_0}{\alpha_1} \right)^{s+1} \). So in both cases \( I_f(u) \rightarrow +\infty \) as \( M_0 \rightarrow +\infty \), which, together with (3.19) shows \( T_1(u) \rightarrow 0 \) as \( M_0 \rightarrow +\infty \), and using the same argument \( T_2(u) \rightarrow 0 \) as \( M_0 \rightarrow +\infty \). This concludes the proof of 3°.

The check of 4° follows the same lines of the previous proof. It suffices to show that there exists \( M_1 > M_0 \) such that if \( \{u_m\} \in E = H_0^1(\Omega), M_1 \leq J_f(u_m) \leq K \) and \( J'_f(u_m)u_m \rightarrow 0 \), then \( \{u_m\} \) is bounded. Let us fix \( m \in \mathbb{N} \) such that \( u_m \in \text{supp} \psi \). Indeed, if \( u_m \notin \text{supp} \psi \), then \( ||u_m|| \) is bounded since the functional \( I_f \) is symmetric and the (PS) condition was proved in [14]. For large \( m \) and \( \rho > 0 \) one has
\[
\rho \|u_m\| + K \geq J_f(u_m) - \rho J'_f(u_m)u_m \\
= \left( \frac{1}{2} - \rho (1 + T_1(u_m)) \right) (\|u_m\|^2 - \lambda \|u_m\|_{L^2}^2) \\
+ [\rho(\psi(u_m) + T_1(u_m)) - \psi(u_m)] \int_{\Omega} fu_m - \int_{\Omega} P(u_m)W(x) \\
+ \rho (1 + T_1(u_m)) \int_{\Omega} u_m p(u_m)W(x) + \rho T_2(u_m) \int_{\Omega} u_m p(u_m)|W(x)|.
\]

For \( M_1 \) sufficiently large, and therefore \( T_1, T_2 \) small, we can choose \( \rho \in ((s + 1)^{-1}, 2^{-1}) \) and \( \varepsilon > 0 \) such that

\[
\frac{1}{2(1 + T_1(u_m))} > \rho + \varepsilon > \rho - \varepsilon > \frac{1}{(s + 1)(1 + T_2(u_m))}.
\]

Hence one finds that

\[
\left( \frac{1}{2} - \rho (1 + T_1(u_m)) \right) (\|u_m\|^2 - \lambda \|u_m\|_{L^2}^2) \geq \frac{\varepsilon}{2} \|u_m\|^2 - C\lambda \|u_m\|_{L^2}^2
\]

and

\[
[\rho(\psi(u_m) + T_1(u_m)) - \psi(u_m)] \int_{\Omega} fu_m \geq -C \|u_m\|_{L^2}^2.
\]

We are going to estimate the last three integrals in (3.21).

\[
- \int_{\Omega} P(u_m)W(x) + \rho (1 + T_1(u_m)) \int_{\Omega} u_m p(u_m)W(x) \\
+ \rho T_2(u_m) \int_{\Omega} u_m p(u_m)|W(x)| \\
\geq - \int_{\Omega} P(u_m)|W(x)| + \rho (1 + T_1(u_m)) \int_{\Omega} u_m p(u_m)W(x) \\
+ \rho T_2(u_m) \int_{\Omega} u_m p(u_m)|W(x)| + \rho \int_{\Omega} u_m p(u_m)|W(x)| \\
- \rho \int_{\Omega} u_m p(u_m)|W(x)| - \rho T_2(u_m)(s + 1) \int_{\Omega} |W(x)|P(u_m) \\
+ \rho (1 + T_2(u_m))(s + 1) \int_{\Omega} |W(x)|P(u_m)
\]
\[= \rho(1 + T_2(u_m)) \int_{\Omega} (u_m p(u_m) - (s + 1) P(u_m)) W(x) \]
\[+ (\rho(1 + T_2(u_m))(s + 1) - 1) \int_{\Omega} |W(x)| P(u_m) \]
\[+ \rho(1 + T_1(u_m)) \int_{\Omega} u_m p(u_m) W(x) - \rho \int_{\Omega} u_m p(u_m) |W(x)|.\]

By (p3) the first integral is positive, while by (2.1) the second integral is greater than
\[- C + \frac{\varepsilon}{2} a_5 \int_{\Omega} |W(x)||u_m|^s+1 \quad \text{(3.26)}\]

For what concerns the third integral one finds, by (p3) and (pW1)
\[\rho(1 + T_1(u_m)) \int_{\Omega} u_m p(u_m)(W^+(x) - W^-(x)) \]
\[\rho(1 + T_1(u_m)) \int_{\Omega} (s + 1) P(u_m) W^+(x) - \gamma \rho(1 + T_1(u_m)) \int_{\Omega} u_m^2 - C \]
\[- \rho(1 + T_1(u_m)) \int_{\Omega} (s + 1) P(u_m) W^-(x) \]
\[\rho(1 + T_1(u_m)) \int_{\Omega} (s + 1) P(u_m) W(x) - C \||u_m||_{L^2}^2 - C \]
\[+ C||u_m||^2 - C \||u_m||_{L^2}^2 - C. \quad \text{(3.27)}\]

It remains to estimate the last integral in (3.25). By (p2) and (p3) one has
\[- \rho \int_{\Omega} u_m p(u_m) |W(x)| \geq -C \int_{\Omega} |u_m| - C \int_{\Omega} P(u_m) |W(x)| - C \]
\[- C \||u_m||_{L^2} - 4AC(|I_f(u_m)|^2 + 1)^{1/2} - C \quad \text{(3.28)}\]

where we have used the fact that \( u_m \in \text{supp} \, \psi \). Let us observe now that
\[I_f(u_m) = J_f(u_m) + (\psi - 1) \int_{\Omega} f u_m \leq K + \left| \int_{\Omega} f u_m \right| \leq K + C \||u_m||_{L^2}^2, \quad \text{(3.29)}\]

hence
\[- \rho \int_{\Omega} u_m p(u_m) |W(x)| \geq -C - C \||u_m||_{L^2}^2. \quad \text{(3.30)}\]
So from (3.21), using (3.23), (3.24), (3.26), (3.27), (3.30), we get
\[ \rho ||u_m|| + K \geq C ||u_m||^2 + C \int_\Omega |W(x)||u_m|^{s+1} - C ||u_m||_2^2 - C. \] (3.31)

Let us observe now that by the Hölder inequality,
\[ ||u||_2^2 = \int_{\text{supp } W} \frac{1}{|W(x)|^{\frac{s+1}{s+2}}} |W(x)|^{\frac{2}{s+2}} |u|^2 \leq C \left( \int_\Omega |W(x)||u|^{s+1} \right)^{\frac{2}{s+2}}. \] (3.32)

By the Young inequality, since \( s+1 > 2 \), for \( \varepsilon > 0 \),
\[ C \left( \int_\Omega |W(x)||u|^{s+1} \right)^{\frac{2}{s+2}} \leq C(\varepsilon) + \varepsilon \int_\Omega |W(x)||u|^{s+1}. \] (3.33)

Choosing \( \varepsilon \) small enough, one finds
\[ \rho ||u_m|| + C \geq C ||u_m||^2 + C \int_\Omega |W(x)||u_m|^{s+1} - C(\varepsilon) - \varepsilon \int_\Omega |W(x)||u|^{s+1}. \]

that is \( \{u_m\} \) is bounded in \( E \). Then by standard arguments 4 holds.  \( \square \)

In order to obtain an unbounded sequence of critical values for the functional \( I_f \), by 3° of Proposition 3.5, it suffices to prove that \( J_f \) possesses multiple critical values. Let us introduce the following minimax values
\[ b_k := \inf_{h \in \Gamma_k} \sup_{u \in E_k} J_f(h(u)) \]
where \( \Gamma_k \) and \( E_k^- \) are defined as in the proof of Theorem 2.2. These minimax values will not in general be critical for \( J_f \) unless \( f \equiv 0 \). Since the presence of the \( \psi \) term in \( J_f \) does not affect the verification of (I1), (I2), arguing as in the proof of Theorem 2.2, we obtain the following lower bound for \( b_k \): there exists a constant \( C \) and \( \tilde{k} \in \mathbb{N} \) such that
\[ b_k \geq C \tilde{k}^\theta \quad \text{for all } k \geq \tilde{k} \] (3.34)
where \( \theta \) was defined in (1.1).

To get critical values of \( J_f \) from the sequence \( \{b_k\} \), another set of minimax values must be introduced. Define
\[ U_k = \{u = t\varphi_{k+1} + w : t \geq 0, w \in E_k^-\} \]
Λ_k = \{ h \in C(U_k, E), \ h \text{ is odd} , h = \text{id on } Q_k \equiv (E^-_{k} \setminus \hat{B}_{R_k}) \cup (E^+_{k+1} \setminus \hat{B}_{R_{k+1}}) \}.

Set
\[ c_k := \inf_{h \in \Lambda_k} \sup_{u \in U_k} J_f(h(u)). \]

Comparing the definition of \( c_k \) and \( b_k \) it is easy to check that \( c_k \geq b_k \).

**Proposition 3.6** Assume \( c_k > b_k \geq M_1 \). For \( \delta \in (0, c_k - b_k) \) define
\[ \Lambda_k(\delta) := \{ h \in \Lambda_k : J_f(h(u)) \leq b_k + \delta \text{ for } u \in E^-_{k} \} \]
and
\[ c_k(\delta) := \inf_{h \in \Lambda_k(\delta)} \sup_{u \in U_k} J_f(h(u)). \] \hspace{1cm} (3.35)

Then \( c_k(\delta) \) is a critical value of \( J_f \).

**Proof**
This proof follows the same lines of Proposition 10.43 of Rabinowitz [18], using 1° and 4° of Proposition 3.5.

Now, if \( c_k > b_k \) for a sequence of \( k \to \infty \), then by Proposition 3.6 and by (3.34), \( J_f \) has an unbounded sequence of critical values and the proof is complete. It remains to show that the relation \( c_k = b_k \) is impossible for all large \( k \).

**Proposition 3.7** If \( c_k = b_k \) for all \( k \geq k^* \), there exists a constant \( C > 0 \), and \( \hat{k} \geq k^* \) such that
\[ b_k \leq Ck^{\frac{d+1}{2}} \] \hspace{1cm} (3.36)
for all \( k > \hat{k} \).

Comparing (3.36) to (3.34) and (1.1), one gets a contradiction and the proof of Theorem 3.2 is complete.

**Proof of Proposition 3.7**
Let \( \varepsilon > 0 \) and \( k \geq k^* \). Choose \( h \in \Lambda_k \) such that
\[ \sup_{u \in U_k} J_f(h(u)) \leq b_k + \varepsilon. \] \hspace{1cm} (3.37)

Since \( U_k \cup (-U_k) = E^-_{k+1} \), \( h \) can be continuously extended to \( E^-_{k+1} \) as an odd function. Therefore, by definition of \( b_k \)
\[ b_{k+1} \leq \sup_{E^-_{k+1}} J_f(h(u)) = \max_{B_{R_{k+1}} \cap E^-_{k+1}} J_f(h(w)) = J_f(h(w)) \] \hspace{1cm} (3.38)
for some \( w \in B_{R_{k+1}} \cap E^-_{k+1} \). If \( w \in U_k \) by (3.37) and (3.38),
\[ J_f(h(w)) \leq b_k + \varepsilon. \] \hspace{1cm} (3.39)
Suppose \( w \in -U_k \). Then, since by \((3.34)\) \( b_k \to \infty \) as \( k \to \infty \), \((3.38)\) and \(2^o\) of Proposition 3.5 imply \( J_f(-h(w)) > 0 \) if \( k \) is large, say \( k \geq \hat{k} \). By \(2^o\), the oddness of \( h \) and \((3.37)\), one gets

\[
J_f(h(w)) = J_f(-h(-w)) \leq J_f(h(-w)) + D \left(\frac{1}{k} \right) + 1
\]

(3.40)

\[
\leq b_k + \varepsilon + D \left( (b_k + \varepsilon) \frac{1}{k} + 1 \right).
\]

Combining \((3.38)-(3.40)\) one finds

\[
b_{k+1} \leq b_k + \varepsilon + D \left( (b_k + \varepsilon) \frac{1}{k} + 1 \right)
\]

(3.41)

Since \( \varepsilon \) is arbitrary, \((3.41)\) implies

\[
b_{k+1} \leq b_k + D \left( (b_k) \frac{1}{k} + 1 \right)
\]

(3.42)

for all \( k \geq \hat{k} \). Arguing as in Proposition 10.46 of Rabinowitz \[18\], \((3.42)\) implies the thesis by induction.

\[\square\]

References


